J. Fluid Mech. (1978), *vol.* 86, *part* 2, *pp.* 273–288
Printed in Great Britain

Inertial waves and an initial-value problem for a thin spherical rotating fluid shell

By J. M. HUTHNANCE
Department of Oceanography, University of Liverpool†

(Received 7 September 1976 and in revised form 16 September 1977)

Natural modes of oscillation of a vanishingly thin spherical rotating fluid shell, with frequencies σ less than twice the angular velocity Ω, were found by Haurwitz (1940). Their validity is, however, put in question by the presence of a singularity at critical co-latitudes θ_c: $2\Omega\cos\theta_c = \sigma$ in the $O(\epsilon)$ term of an expansion in the relative shell thickness ϵ (Stewartson & Rickard 1969). The problem is investigated here by considering the evolution of flow from a specified initial distribution. The principal features are as follows:

(i) The $O(1)$ natural modes of Haurwitz, decaying on a time scale $\Omega^{-1}\epsilon^{-2}$.

(ii) Corrections to (i), regular and of magnitude $O(\epsilon)$ except near critical latitudes.

(iii) Essentially transient inertial waves of magnitude $O(\epsilon)$.

(iv) Inertial waves of magnitude $O(\epsilon^{\frac{1}{2}})$ with the natural-mode frequencies σ and generated by (i) at critical latitudes.

On a time scale $\Omega^{-1}\epsilon^{-1}$, (iii) and (iv) develop vertical and *horizontal* length scales ϵ and propagate throughout the ocean. The continuing energy transfer from (i) to (iv), at a rate $O(\epsilon^2\Omega)$, appears to be the principal respect in which (i) and (ii) fail to constitute a conventional normal mode.

1. Introduction

The rotating thin spherical shell of homogeneous inviscid fluid has attracted considerable historical interest as the simplest model for the oceans, particularly in connexion with tides. For a shell defined by concentric spheres with radii $a > b$, a formal expansion of the equations of motion in the relative shell thickness $\epsilon = \log(a/b)$ yields (Pekeris 1975) Laplace's tidal equations (henceforth LTE) as the largest, $O(\epsilon^0)$ terms. LTE are used extensively for the study of long waves on the ocean.

Taking the sea surface (radius a) as rigid corresponds to the special case of a zero divergence parameter $4\Omega^2a^2/g(a-b)$. This assumption corresponds to large gravity (for example), and is appropriate when LTE are applied to a small region of the earth's oceans. In this case, Haurwitz (1940) and Longuet-Higgins (1964) found natural modes for the complete spherical shell. They have zero radial velocity and a meridional velocity in the form of tesseral harmonics. However, the validity of these solutions of LTE is in doubt.

† Present address: Institute of Oceanographic Sciences, Bidston Observatory, Birkenhead, Merseyside L43 7RA, England.

Stewartson & Rickard (1969, henceforth denoted by SR) attempted to find natural modes for a thin spherical shell ($1 \gg \epsilon \neq 0$) by using the complete equations of motion and expanding in ϵ about the $O(\epsilon^0)$ tesseral-harmonic solution. This procedure contrasts with taking $\epsilon = 0$ *a priori* by use of LTE. The $O(\epsilon)$ term of the expansion was found to be singular at critical co-latitudes θ_c: $\cos\theta_c = \pm\sigma/2\Omega$, where the natural-mode frequency σ equals the local inertial frequency, viz. twice the vertical component of the earth's rotation. By the addition of $O(\epsilon^{\frac{1}{3}})$ motion having fine structure (vertical and horizontal length scales ϵa), the severity of the singularity was reduced to degree $-\frac{1}{2}$ for the velocity. This is integrable to a continuous pressure distribution, but not integrable in square for kinetic energy. Subsequently, Stewartson & Walton (1976, henceforth denoted by SW) were able to complete this fine-structure solution around the sphere without further singularities, although the velocity singularity continues all the way round on reflected characteristics.

The difficulty at critical latitudes reflects the ill-posed nature of the mathematical problem governing natural modes of frequency $\sigma < 2\Omega$, namely

$$\left.\begin{array}{ll} & (\sigma/2\Omega)^2\nabla^2 p - \partial^2 p/\partial z^2 = 0 \\ \text{with} & \text{zero radial velocity on } r = a, b. \end{array}\right\} \tag{1.1}$$

Problem (1.1) is hyperbolic, but has boundary conditions suitable for an elliptic equation. By contrast, LTE do yield an elliptic field equation:

$$\nabla_H \cdot [(\sigma^2 - 4\Omega^2\cos^2\theta)^{-1}\nabla_H p] = f(p, \nabla_H p),$$

where ∇_H is the horizontal gradient operator.

In physical terms, the radial momentum equation contains a Coriolis force term (neglected in LTE) associated with the local horizontal component of the earth's rotation. In a homogeneous fluid this can be balanced only by an $O(1)$ radial pressure gradient. Hence there are $O(\epsilon)$ pressure corrections of frequency σ which will drive inertial motions resonantly at critical latitudes.

The introduction of stratification, viscosity or an initial-value approach may be expected to determine smooth solutions. The first two (and especially stratification) might be regarded from a mathematical viewpoint as changes in the problem, but are nevertheless important in the ocean.

Miles (1974) has concluded that LTE form a valid description of barotropic motion if the buoyancy frequency N (a measure of the stratification) greatly exceeds 2Ω as is usual in the ocean. Buoyancy forces then assist in balancing the radial Coriolis force, removing the critical-latitude singularity. SW considered the normal-mode problem more specifically, for various degrees of stratification as measured by $N^2/(2\Omega)^2$ relative to ϵ. There is no essential change from the homogeneous case until

$$N^2/(2\Omega)^2 \gtrsim 1,$$

when the replacement for (1.1) is hyperbolic only between co-latitudes $\pm\theta_B$:

$$\cos^2\theta_B = \frac{\sigma^2}{4\Omega^2}\left(\frac{4\Omega^2 - \sigma^2}{N^2} + 1\right).$$

The fine structure is thus confined between $\pm\theta_B$, which close from the poles to the critical co-latitudes as $N^2/(2\Omega)^2$ increases to $O(\epsilon^{-\frac{1}{3}})$, when the critical-latitude singularities are removed.

Walton (1975) has shown that a small kinematic viscosity ν ($\epsilon^3 \gg E \equiv 2\nu/(\Omega a^2\epsilon^2)$) reduces the (degree $-\frac{1}{2}$) velocity singularities of the homogeneous case to smooth shear layers of breadth $a\epsilon E^{\frac{1}{3}}$ and maximum velocity $O(\epsilon^{\frac{1}{2}}E^{-\frac{1}{6}})$. Energy is lost here at a rate $O(\Omega\epsilon E^{\frac{1}{3}})$ relative to the total in the natural mode. The shear layers, along characteristics of (1.1), further suffer a fractional loss $O(E^{\frac{1}{3}})$ of intensity at successive reflexions off the bounding shells away from critical latitudes. In practice this means effective confinement of the fine structure close to critical latitudes. The introduction of viscosity generally appears to support the SR solution, including the fine structure, as a valid normal mode.

Normal modes are of interest in facilitating the solution of initial-value or forced-oscillation problems. Their validity therefore depends on how accurately they may represent the solutions of such problems. An initial-value problem is considered here; this also sheds some light on forced oscillations. By ignoring stratification and viscosity, we are attempting to understand better the basic homogeneous, inviscid system, and in particular why the normal-mode problem is mathematically ill posed and unsatisfactory in practice, rather than attempting to describe the real ocean which inspired this model.

Initially the velocity is specified to have zero radial component, and meridional and zonal components corresponding to one of the Haurwitz normal-mode solutions of LTE. Since tesseral harmonics are complete on the spherical surface, no loss of generality is implied among initial current distributions of global length scale. The full equations of motion and the initial time development are described in § 2. The initial conditions result in essentially transient $O(\epsilon)$ inertial waves whose evolution on the time scale $\Omega^{-1}\epsilon^{-1}$ is followed in § 3. Of more significance for the normal-mode question are the $O(\epsilon^{\frac{1}{2}})$ inertial waves generated at critical latitudes, which are treated separately in § 4.

2. Equations of motion and initial solution

We consider a homogeneous, inviscid, incompressible fluid with small amplitude motion $\mathbf{u} \equiv (u, v, w)$, in spherical polar co-ordinates (θ, ϕ, r), relative to a reference frame rotating with angular velocity $\mathbf{\Omega}$. The linearized equations of motion are

$$\left.\begin{aligned} \partial\mathbf{u}/\partial t + 2\mathbf{\Omega}\wedge\mathbf{u} &= -\nabla p, \\ \nabla.\mathbf{u} &= 0, \end{aligned}\right\} \qquad (2.1)$$

where $p = \rho^{-1}(\text{pressure}) - \frac{1}{2}\Omega^2 r^2 \sin^2\theta.$

For a constituent harmonic in ϕ, the normalization (Miles, private communication)

$$(u, v, w) = (2\Omega a^2/r\mu_*)\,\mathrm{Re}\,[(U, iV, \epsilon W)\,e^{im\phi}],$$

$$p = (2\Omega a)^2\,\mathrm{Re}\,[iP\,e^{im\phi}]$$

transforms (2.1) to

$$\begin{bmatrix} -i\,\partial/\partial T & -\mu & 0 & -D \\ -\mu & -i\,\partial/\partial T & -\epsilon\mu_* & m \\ 0 & -\epsilon\mu_* & -i\epsilon^2\,\partial/\partial T & \mu_*\,\partial/\partial\xi \\ D & m & -\mu_*(\partial/\partial\xi+\epsilon) & 0 \end{bmatrix} \begin{pmatrix} U \\ V \\ W \\ P \end{pmatrix} = 0. \qquad (2.2\,a\text{–}d)$$

Here $\mu = \cos\theta, \mu_* = \sin\theta, \xi = \epsilon^{-1}\log r/a, D = \mu_*^2\,\partial/\partial\mu, T = 2\Omega t$ and $\mathbf{U} \equiv (U, V, W, P)$ is a function of μ, ξ and T. The motion is also subject to boundary conditions

$$\left.\begin{aligned} &\text{radial velocity} \quad W = 0 \quad \text{on} \quad \xi = -1, 0 \quad \text{(bounding spheres)}, \\ &\mathbf{U} \quad \text{bounded at} \quad \mu = \pm 1 \quad \text{(poles)} \end{aligned}\right\} \tag{2.3}$$

and initial conditions

$$(U, V, W)(\mu, \xi, 0) = (m, -D, 0)P_n^m(\mu) \quad (n \geqslant |m| > 0). \tag{2.4}$$

This form of initial flow field, independent of the vertical co-ordinate ξ, corresponds to one of the Haurwitz (1940) normal-mode solutions of (2.2) with $\epsilon = 0$ (i.e. LTE), namely

$$\mathbf{U} = \mathbf{U}_0 \equiv (m, -D, 0, \mu + \lambda D/m)P_n^m(\mu)\,e^{i\lambda T}, \tag{2.5}$$

where $\lambda = m/n(n+1)$ and the associated Legendre function $P_n^m(\mu)$ satisfies

$$\frac{\partial}{\partial\mu}\left(\mu_*^2\frac{\partial P_n^m}{\partial\mu}\right) + \left(\frac{m}{\lambda} - \frac{m^2}{\mu_*^2}\right)P_n^m = 0.$$

Any initial flow field which varies only on global length scales (and is therefore independent of ξ) may be expressed as a combination of forms (2.4) for various m and n, apart from axisymmetric constituents ($m = 0$). The latter have $O(\epsilon^{\frac{1}{2}})$ normal-mode frequencies and are subject to separate consideration (Stewartson 1971), the critical latitudes being close to the equator. Hence the use of (2.4) involves no loss of generality in the present context.

If ϵ is neglected entirely, the Haurwitz normal mode (2.5) solves (2.2)–(2.4) exactly. However, our purpose is to investigate the small-ϵ corrections which appear to be singular for the normal modes (2.5). Hence we let

$$\mathbf{U} = \mathbf{U}_0 + \epsilon\mathbf{U}_1(\mu, \xi, T) + \dots,$$

$$\mathbf{L}_0 = \begin{bmatrix} -i\,\partial/\partial T & -\mu & 0 & -D \\ -\mu & -i\,\partial/\partial T & 0 & m \\ 0 & 0 & 0 & \mu_*\,\partial/\partial\xi \\ D & m & -\mu_*\,\partial/\partial\xi & 0 \end{bmatrix}, \quad \mathbf{L}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\mu^* & 0 \\ 0 & -\mu_* & 0 & 0 \\ 0 & 0 & -\mu_* & 0 \end{bmatrix}$$

so that the terms of (2.2) of order $\epsilon^0, \epsilon^1, \dots$ are

$$\mathbf{L}_0\mathbf{U}_0 = 0, \tag*{$(2.6)_0$}$$

$$\mathbf{L}_0\mathbf{U}_1 = -\mathbf{L}_1\mathbf{U}_0 = (0, 0, \mu_* V_0, 0) \tag*{$(2.6)_1$}$$

and so forth. Equation $(2.6)_0$ is already satisfied by the choice (2.5) of $\mathbf{U}_0$. Equation $(2.6)_1$, together with the boundary and initial conditions (2.3) and (2.4), then determines the corrections $\epsilon\mathbf{U}_1$ in the form

$$P_1 = V_0(\xi + \tfrac{1}{2})\,e^{i\lambda T},$$

$$(U_1, V_1, W_1) = \mathbf{X}_\lambda e^{i\lambda T} + \mathbf{X}_+ e^{i\mu T} + \mathbf{X}_- e^{-i\mu T}.$$

This correction $\epsilon\mathbf{U}_1$ applies to the whole sphere including critical latitudes. The three-component vectors $\mathbf{X}_\lambda$ and $\mathbf{X}_\pm$ are specified in table 1. $\mathbf{X}_\lambda$ represents the $O(\epsilon)$ correction to the normal mode, arising as a particular solution of the forced equation $(2.6)_1$. The oscillations $\mathbf{X}_\pm e^{\pm i\mu T}$ at the local inertial frequencies $\pm\mu$ are complementary functions (satisfying the homogeneous form of $(2.6)_1$) introduced to satisfy the initial conditions.

From ... / To	$\mathbf{X}_\lambda$	$\mathbf{X}_+$	$\mathbf{X}_-$
$\dfrac{\mu^2-\lambda^2}{m(\xi+\frac{1}{2})} U_1$	$(m\lambda-\mu_*^2-\mu D)\, P_n^m$	$-\dfrac{\lambda+\mu}{2} U_+$	$-\dfrac{\lambda-\mu}{2} U_-$
$\dfrac{\mu^2-\lambda^2}{m(\xi+\frac{1}{2})} V_1$	$\left(m\mu-\dfrac{\mu\mu_*^2}{\lambda}-\lambda D\right) P_n^m$	$-\dfrac{\lambda+\mu}{2} U_+$	$+\dfrac{\lambda-\mu}{2} U_-$
$\dfrac{4(\mu^2-\lambda^2)^2}{\mu_* m\xi(\xi+1)} W_1$	$[4\lambda^2 D+4\mu(1-\lambda^2-m\lambda)]\, P_n^m$	$-(\lambda+\mu)^2\left[\left(2-\dfrac{\mu}{\lambda}\right)D + \dfrac{1+\mu^2}{\lambda}-m-2\mu\right] P_n^m - (\lambda+\mu)^2(\mu-\lambda)\, iTU_+$	$-(\lambda-\mu)^2\left[\left(2-\dfrac{\mu}{\lambda}\right)D - \dfrac{1+\mu^2}{\lambda}+m-2\mu\right] P_n^m - (\lambda-\mu)^2(\lambda+\mu)\, iTU_-$

TABLE 1. Contributions to $O(\epsilon)$ terms in the initial expansion. Here $U_\pm = (m-\mu_*^2/\lambda \mp D)\, P_n^m$.

As indicated by SR(2.17), $\mathbf{X}_\lambda$ is singular at $\mu = \pm\lambda$. For finite times, however, these singularities are an artifact of the division of $\mathbf{U}_1$ into normal-mode and inertial terms. $\mathbf{X}_\pm$ has an equal compensating singularity at $\mu = \pm\lambda$, so that $\mathbf{U}_1$ is regular, as expected for the initial-value problem (note that $e^{\pm i\mu T}$ tends to $e^{i\lambda T}$ as $\mu \to \pm\lambda$). Nevertheless, as $\mu \to \pm\lambda$, factors (T, T, T^2) appear in some terms for (U_1, V_1, W_1), so that $\mathbf{U}_1$ becomes large near critical latitudes as time passes. In fact, for large times T the expansion breaks down in three ways:

(i) W_1 contains a factor T and becomes large.

(ii) Continuation of the expansion in ϵ leads to apparent factors

$$(\mu^2-\lambda^2)^{-2n+(1,1,0,3)}$$

in (U_n, V_n, W_n, P_n), as indicated by SR(3.1) for the normal-mode terms. Powers $2n-(1,1,0,3)$ of T at critical latitudes (rather than singularities) result when the inertial terms are included.

(iii) Throughout the fluid, the inertial-wave parts of (U_n, V_n, W_n, P_n) have explicit factors $T^{2n-(2,2,1,4)}$ $(n \geqslant 2)$.

Hence the expansion in ϵ is valid only while $T \ll \epsilon^{-\frac{1}{2}}$ and successive terms of size $\epsilon^n T^{2n+q}$ decrease with n. In the following section we consider the later development of the inertial waves $\mathbf{X}_\pm e^{\pm i\mu T}$, excluding the spurious initial singularities at critical latitudes. $\Omega^{-1}\epsilon^{-1}$ emerges as the principal time scale for inertial-wave development. Although (in the absence of viscosity) their energy does not decay, those inertial waves $\mathbf{X}_\pm e^{\pm i\mu T}$ not initially close to critical latitudes are essentially transient, being subsequently uncoupled with the normal-mode oscillation of frequency λ. Hence §3 does not bear directly on the normal-mode question, to which we return in §4 with the separate treatment of critical latitudes.

3. Transient inertial waves

3.1. *Dispersion relation*

The terms $\mathbf{X}_\pm e^{\pm i\mu T}$ of the initial solution arose from the application of the initial conditions at order ϵ. Apart from the neighbourhood of critical latitudes, where

$\mu \simeq \pm\lambda$, these inertial waves must subsequently satisfy the equations of motion (2.2) and boundary conditions (2.3) independently. The form of the $T = O(1)$ solution suggests that a rescaled vertical velocity $\epsilon^{\frac{1}{2}}W_1$ should be introduced to seek a new form of solution $\mathbf{X}_{\pm}(\mu, \xi, \epsilon^{\frac{1}{2}}T)\, e^{\pm i\mu T}$ at times $O(\Omega^{-1}\epsilon^{-\frac{1}{2}})$. This may be substituted in (2.2) and a solution obtained by decomposition into vertical structure modes $e^{2\pi n i\xi}$ ($e^{2\pi n i \xi}-1$ for $\epsilon^{\frac{1}{2}}W_1$). However, this also breaks down, when $T = O(\epsilon^{-1})$, in two respects:

(i) $\epsilon^{\frac{1}{2}}W_1$ has further grown to $O(\epsilon^{-\frac{1}{2}})$;

(ii) $\mathbf{X}_{\pm}$ varies rapidly, at a rate $O(\epsilon^{-\frac{1}{2}}, \epsilon^{-1})$, with $\epsilon^{\frac{1}{2}}T$ and μ.

It is therefore necessary to reconsider the solution for $T = O(\epsilon^{-1})$; the result covers $T = O(\epsilon^{-\frac{1}{2}})$ in any case.

We write $\tau = \epsilon T$, and seek a solution of (2.2) in the form

$$(U, V, \epsilon W, \epsilon^{-1}P) = \epsilon[\mathbf{U}^{(1)} + \epsilon\mathbf{U}^{(2)} + \ldots](\mu, \xi, \tau)\exp[i\phi(\mu, \tau)/\epsilon], \tag{3.1}$$

which is to correspond to the vertical structure $e^{2\pi n i\xi}$ when $T = O(\epsilon^{-\frac{1}{2}})$. The phase function ϕ yields a local meridional wavenumber and frequency

$$l(\mu, \tau) \equiv \mu_* \,\partial\phi/\partial\mu, \quad \sigma(\mu, \tau) \equiv \partial\phi/\partial\tau$$

for oscillations on a length scale ϵa and time scale Ω^{-1}. We also write $\sigma_* = (1-\sigma^2)^{\frac{1}{2}}$. Substitution of (3.1) in (2.2) yields

$$\mathbf{A}(\mathbf{U}^{(1)} + \epsilon\mathbf{U}^{(2)} + \ldots) = \epsilon\mathbf{B}\mathbf{U}^{(1)} + \ldots, \tag{3.2}$$

where

$$\mathbf{A} = \begin{bmatrix} \sigma & -\mu & 0 & -i\mu_* l \\ -\mu & \sigma & -\mu_* & 0 \\ 0 & -\mu_* & \sigma & \mu_*\,\partial/\partial\xi \\ i\mu_* l & 0 & -\mu_*\,\partial/\partial\xi & 0 \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} i\,\partial/\partial\tau & 0 & 0 & D \\ 0 & i\,\partial/\partial\tau & 0 & -m \\ 0 & 0 & i\,\partial/\partial\tau & 0 \\ -D & -m & \mu_* & 0 \end{bmatrix}.$$

The lowest-order equations $\mathbf{A}\mathbf{U}^{(1)} = 0$ must be satisfied first. They determine $U^{(1)}$, $V^{(1)}$ and $P^{(1)}$ successively in terms of $W^{(1)}$, which satisfies a second-order differential equation in ξ (only), with coefficients independent of ξ. Thus

$$W^{(1)} = W(\mu, \tau)[\exp(im_+\xi) - \exp(im_-\xi)], \tag{3.3}$$

where

$$m_{\pm} = l(\mu\mu_* \pm \sigma\sigma_*)/(\sigma^2 - \mu^2)$$

and the boundary condition $W^{(1)} = 0$ on $\xi = 0$ has been incorporated. We may identify $m_{\pm}$ with the roots of the usual dispersion relation

$$\sigma^2 = (l\mu_* + m\mu)^2/(l^2 + m^2)$$

for inertial waves. Thus the lowest-order terms of (3.2) simply require that the free oscillations arising from the initial conditions satisfy the local inertial-wave dispersion relation.

3.2. *Evolution*

The final boundary condition $W^{(1)} = 0$ on $\xi = -1$ requires

$$\pm 2l\sigma\sigma_*/(\sigma^2 - \mu^2) = \pm(m_+ - m_-) = 2n\pi, \tag{3.4}$$

where n is an integer which we identify as the vertical structure mode number when $T = O(\epsilon^{-\frac{1}{2}})$. Thus the vertical wavenumbers $m_{\pm}$ are quantized and the phase function ϕ is constrained by (3.4). Solving (3.4) for $l = -\partial\phi/\partial\theta$ and differentiating with respect to τ gives a first-order linear partial differential equation for $\sigma = \partial\phi/\partial\tau$. The solution is

$$\sigma = \text{constant} = \pm\cos\theta_0$$

on the characteristics $\theta(\tau, \theta_0)$ defined by $[\partial\theta/\partial\tau]_{\theta_0}$ = group velocity $= -\partial\sigma/\partial l$, namely

$$0 = F(\tau, \theta_0, \theta) \equiv (\tau/|n|\pi)\sin\theta_0 \sin^2 2\theta_0 + S\{2(\theta_0-\theta)+\cos 2\theta_0(\sin 2\theta - \sin 2\theta_0)\}. \tag{3.5}$$

Here θ_0 is the co-latitude of the characteristic initially (when $\tau = 0$), when the frequency σ of the inertial-wave transient was $\pm\mu = \pm\cos\theta_0$. S is the sign of $\pm n$ in the case of an initial frequency $\pm\mu$. It should perhaps be emphasized that these characteristics in θ, τ space are quite unrelated to those of the normal-mode problem in r, θ space.

The form of the slowly varying wave amplitude $W(\mu, \tau)$ in (3.3) is determined by the $O(\epsilon)$ terms of (3.2):

$$\mathbf{A}\mathbf{U}^{(2)} = \mathbf{B}\mathbf{U}^{(1)}. \tag{3.6}$$

By analogy with the lowest-order terms, $U^{(2)}$, $V^{(2)}$ and $P^{(2)}$ may be found successively in terms of $W^{(2)}$ and $\mathbf{U}^{(1)}$. $W^{(2)}$ then satisfies an inhomogeneous form of the equation satisfied by $W^{(1)}$. Thus we have a linear system forced by $\mathbf{U}^{(1)}$ (which may be expressed in terms of $W^{(1)}$) but possessing a non-trivial solution when unforced. The requirement that a solution $W^{(2)}$ exists imposes conditions on the forcing, i.e. on $W(\mu, \tau)$. An equivalent alternative approach is to form an energy balance from $\mathbf{A}\mathbf{U}^{(1)} = 0$ and (3.6), namely

$$\int_{-1}^{0} \{\mathbf{U}^{(2)}.(\mathbf{A}\mathbf{U}^{(1)})^* + \mathbf{U}^{(1)*}.(\mathbf{B}\mathbf{U}^{(1)} - \mathbf{A}\mathbf{U}^{(2)})\}\,d\xi = 0, \tag{3.7}$$

where the superscript $*$ denotes complex conjugation. On evaluation and use of the boundary conditions $W^{(1)} = 0 = W^{(2)}$ on $\xi = -1, 0$, the imaginary part is found to be

$$\frac{\partial}{\partial\tau}\int_{-1}^{0} \tfrac{1}{2}(|U^{(1)}|^2 + |V^{(1)}|^2 + |W^{(1)}|^2)\,d\xi + D\int_{-1}^{0} \operatorname{Im} P^{(1)}U^{(1)*}\,d\xi + \mu_*\int_{-1}^{0} \operatorname{Im} P^{(1)*}W^{(1)}\,d\xi = 0. \tag{3.8}$$

The last term is in fact zero, so that the local increase of kinetic energy in the evolving transient inertial wave is due simply to convergence of its meridional energy flux. The latter may be evaluated (in dimensional form) as

$$\int_a^b \rho\,\overline{pu}\,dr = \frac{\rho\epsilon a}{2\mu_*}(2\Omega\epsilon a)^3\left\{-\operatorname{Im}\int_{-1}^{0} P^{(1)}U^{(1)*}\,d\xi = \frac{2|n|\pi S\sigma_*}{l^2\mu_*}|W|^2\right\}.$$

In particular, the energy flux is southward or northward according to the sign of S [i.e. according to the progress $d\theta/d\tau$ of the characteristics (3.5)].

All quantities in (3.8) are expressible in terms of $W^{(1)}$, so that an evolution equation for $|W|^2$ may be found. Similarly, the real part of (3.7) determines the evolution of arg W. The results are simpler in characteristic co-ordinates (τ, θ_0):

$$\mu_*^{-1}(\mu^2-\sigma^2)^{-2}\frac{\partial F}{\partial\theta_0}|W|^2, \quad 2\arg W - \frac{\mu\mu_* l}{\sigma^2-\mu^2}$$

are constant on characteristics. W and hence $\mathbf{U}^{(1)}$ are thus determined by the initial condition

$$W \sim \frac{im\mu_*\tau U_\pm}{8n^2\pi^2(\lambda\mp\mu)} \quad \text{as} \quad \tau\to 0$$

obtained from the initial solution at large T. Other initial conditions $W \sim W_0(\mu; n)$ representing any initial distribution of inertial waves may be accommodated by this solution.

Thus the transient inertial-wave motion evolves along the group-velocity characteristics (3.5), in the sense of both maintaining constant frequency and energy conservation. The characteristics are sketched in figure 1 for the case $S = 1$; $S = -1$ yields a mirror image in $\theta = \frac{1}{2}\pi$. When $S = 1$ it is easily shown that the characteristics have positive slope, and it is also possible to show, as suggested by the figure, that

(i) characteristics starting in $\theta_0 < \frac{1}{2}\pi$ do not meet in $\theta < \frac{1}{2}\pi$;

(ii) characteristics starting in $\theta_0 > \frac{1}{2}\pi$ do not meet.

To a large extent, therefore, the transformation from (θ, τ) to characteristic co-ordinates (θ_0, τ) is non-singular. The Jacobian is

$$\frac{\partial(\theta, \tau)}{\partial(\theta_0, \tau)} = -\frac{\partial F}{\partial \theta_0}\Big/\frac{\partial F}{\partial \theta},$$

which is the source of the factor $(\partial F/\partial\theta_0)^{-\frac{1}{2}}$ in the amplitude of W. As the characteristics crowd closer together, so that the Jacobian (i.e. $\partial F/\partial\theta_0$) is small, the wave energy is compressed, so that $|W|^2$ increases as $(\partial F/\partial\theta_0)^{-1}$.

3.3. *Further developments*

We remark that the linearity of the problem implies a simple addition of solutions where characteristics cross. Then the above solution for the evolving transient inertial waves remains valid indefinitely, except perhaps along singularities of the transformation to characteristic co-ordinates, at latitudes where $\mu = \pm\sigma$ and at the poles $\mu = \pm 1$. We discuss these briefly in turn.

It is clear from figure 1 (in which $S = +1$) that the characteristics from $\theta_0 < \frac{1}{2}\pi$ form a caustic in $\theta > \frac{1}{2}\pi$ where the Jacobian $\partial(\theta, \tau)/\partial(\theta_0, \tau)$ vanishes. It may be shown, as the figure suggests, that as time progresses characteristics from decreasing $\theta_0 < \frac{1}{2}\pi$ successively arrive at this envelope or wave front, which we denote by $\theta_E(\tau)$, and are subsequently overtaken by a characteristic of lesser θ_0. Since the wave energy at any frequency $\cos\theta_0$ follows the characteristic labelled by θ_0, this implies a steadily increasing frequency at the wave front. Close to the caustic $\theta_E(\tau)$, we may expect an increased wave energy density associated with the crowding of characteristics; this also implies a more rapid spatial variation of $W(\mu, \tau)$ as these wave amplitude functions having different histories arrive in close proximity. Accordingly, it is appropriate to seek a solution of the form

$$(U, V, \epsilon W, \epsilon^{-1}P) = \epsilon^{1-\frac{1}{2}p}[\mathbf{U}^{(1)} + \epsilon^q\mathbf{U}^{(2)} + \dots](y, \xi, \tau)\exp[i\phi(\mu, \tau)/\epsilon],$$

where y is the magnified spatial co-ordinate $\epsilon^{-p}[\theta - \theta_E(\tau)]$, ϕ takes its known value

$$\int_0^\tau \left(\sigma_E - l_E\frac{d\theta_E}{d\tau'}\right)d\tau' - l_E\epsilon^p y$$

near $\theta_E(\tau)$, and σ_E and l_E are the known values of σ and l on $\theta_E(\tau)$. After some investigation of the balance of terms at successive orders of approximation ϵ^0, ϵ^q and ϵ^{2q}, it appears that appropriate choices are $p = \frac{2}{3}$ and $q = \frac{1}{3}$. Thus the wave front has a breadth $\epsilon^{\frac{2}{3}}$ and the wave amplitudes there are $\epsilon^{\frac{2}{3}}$ compared with ϵ elsewhere. The

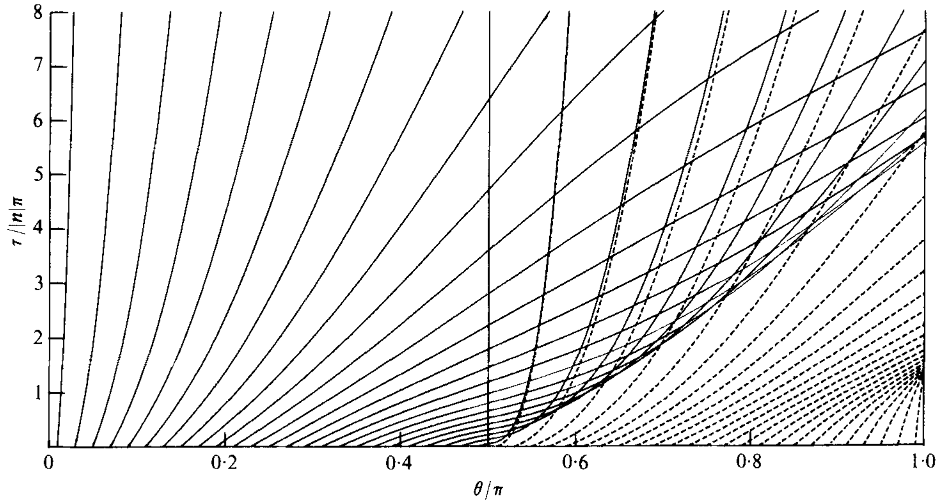

FIGURE 1. Characteristics (3.5) for the propagation of the nth-mode inertial wave over the sphere. Characteristics originating in $\theta > \frac{1}{2}\pi$ are drawn dashed for clarity.

wave envelope is represented by an Airy function (Abramowitz & Stegun 1965, p. 446):

$$W \propto \mathrm{Ai}\,(|a|^{\frac{1}{3}}y),$$

where $a(\tau)$ is a bounded function of time along the wave front $\theta_E(\tau)$. This solution gives exponential decay ahead of the wave front, and its asymptotic form behind, as $y \to -\infty$, matches the previous form of $W(\mu, \tau)$ as implied by its development along characteristics, except that the phase of W on the characteristic emerging from the front is advanced by $\frac{1}{2}\pi$.

At the latitudes $\mu = \pm\sigma$ where the local inertial frequency equals that of the traveling inertial wave, the wave motion becomes horizontal and the wavenumber vector is directed along the local vertical. Thus the phase of W remains as defined in § 3.2, but $W \to 0$, so that the amplitude of the motion might be better described by $U^{(1)}(\mu, \xi, \tau)$ (say). Otherwise there is a smooth transition through these latitudes.

The characteristics in figure 1 ($S = +1$) eventually reach the south pole $\mu = -1$, and likewise $S = -1$ characteristics with their associated wave energy arrive at the north pole $\mu = +1$. For the case illustrated by the figure, the first arrivals are at $\tau = \frac{4}{3}|n|\,\pi$, when there is a concentration at the south pole of characteristics originating from nearby. Strong and probably complex inertial wave motion is then to be expected; the form (3.1) which develops along characteristics is in any case inappropriate near poles, where the first two rows of the governing matrix $\mathbf{A}$ become linearly dependent. New characteristics, along which the inertial wave energy may propagate away from the polar region, may be expected to start at $\tau = \frac{4}{3}|n|\,\pi$. The energy involved, originating from within a distance $O(\epsilon^{\frac{1}{2}})$ of the pole, is only a small, $O(\epsilon)$ fraction of the total in the transient inertial motion.

Subsequently characteristics arrive from non-polar regions. Inspection of the form (3.1) appropriate elsewhere suggests that a polar form

$$(U, V, \epsilon W, P) = \epsilon^{\frac{3}{4}}\mathbf{U}^{(1)}(y, \tau, \xi)\, e^{i\sigma T}$$

should be sought, the ξ dependence being $e^{in\pi\xi} \pm e^{-in\pi\xi}$ (with the minus sign for $W^{(1)}$ only). Here $y = \epsilon^{-1}\mu_*$ is a magnified radial distance from the pole; the implied horizontal length scale is ϵ. Substitution in (2.2) yields

$$P^{(1)}(y,\tau) = P(\tau)\,J_m(-ly),$$

from which the velocity components follow. J_m is a Bessel function of degree m, and its asymptotic form for large y,

$$y^{-\frac{1}{2}}\{\exp(-ily)+\exp[ily+i(m+\tfrac{1}{2})\pi]\},$$

matches the solution on the arriving characteristic. There is also a 'reflected' wave with a phase change $(m+\frac{1}{2})\pi$ which will propagate along a characteristic characterized by σ and leaving the pole at the time of arrival. Alternatively, one may regard the arriving characteristic, with its associated waves, as simply continuing across the pole with a phase change $\frac{1}{2}\pi$. This solution and phase change at the pole are analogous to the components SW(6.17) of the SR normal-mode fine structure.

4. Critical latitudes

In the discussion of the transient inertial waves in §3, we excluded those waves starting near $\mu = \pm\lambda$, where $\mathbf{X}_\pm$ is singular. In the neighbourhood of $\mu = +\lambda$ (say, taking this particular case for definiteness), the development of the singular $O(\epsilon)$ normal-mode term $\mathbf{X}_\lambda\, e^{i\lambda T}$ and the inertial term $\mathbf{X}_+\, e^{i\mu T}$, with its cancelling singularity in the initial solution, must be considered together. The form of the initial solution indicates that an $\epsilon^{-\frac{1}{2}}$ neighbourhood of $\mu = \lambda$ should be considered; as concluded in §2, fresh consideration is required after a time $O(\Omega^{-1}\epsilon^{-\frac{1}{2}})$, when

$$(U_1, V_1, W_1, P_1) = O(\epsilon^{-\frac{1}{2}}, \epsilon^{-\frac{1}{2}}, \epsilon^{-1}, 1).$$

4.1. *Times* $O(\Omega^{-1}\epsilon^{-\frac{1}{2}})$

We write $\delta = \epsilon^{\frac{1}{2}}$, $s = \delta\lambda_* T$ and $x = (\mu-\lambda)/\lambda_*\delta$, where $\lambda_* = (1-\lambda^2)^{\frac{1}{2}}$, and seek

$$\mathbf{U} = \mathbf{U}_0 + (0,0,0,\epsilon P_1)\,e^{i\lambda T} + (\delta\overline{u}, \delta\overline{v}, \overline{w}, \delta^3\overline{p})\,(s,x,\xi)\,e^{i\lambda T},$$

where $P_1 = (\xi+\frac{1}{2})\,V_0|_{\mu=\lambda}$ balances the vertical Coriolis force as before.

The fine structure $\overline{\mathbf{u}}$ is of the form considered by SR with the addition of slow time dependence. Substitution in (2.2) yields

$$\begin{bmatrix} \lambda - i\delta\lambda_*\,\partial/\partial s & -(\lambda+\delta\lambda_* x) & 0 & -\delta\lambda_*\,\partial/\partial x \\ -(\lambda+\delta\lambda_* x) & \lambda - i\delta\lambda_*\,\partial/\partial s & -\delta\lambda_* & 0 \\ 0 & -\lambda_* & 0 & \lambda_*\,\partial/\partial\xi \\ \lambda_*\,\partial/\partial x & 0 & -\lambda_*\,\partial/\partial\xi & 0 \end{bmatrix} \begin{bmatrix} \overline{u} \\ \overline{v} \\ \overline{w} \\ \overline{p} \end{bmatrix} = \delta \begin{bmatrix} DP_1 \\ -mP_1 \\ 0 \\ 0 \end{bmatrix}_{\mu=\lambda} + O\begin{bmatrix} \delta^2 \\ \delta^2 \\ \delta \\ \delta \end{bmatrix}.$$

Hence to lowest order in δ, $\overline{u} = \overline{v} = \partial\overline{p}/\partial\xi$, $\partial\overline{w}/\partial\xi = \partial\overline{u}/\partial x$ and

$$\left(i\frac{\partial}{\partial s}+x\right)(\overline{u}+\overline{v}) + \overline{w} + \frac{\partial}{\partial x}\overline{p} = -\left(\frac{D-m}{\lambda_*}\right)P_1 = \frac{m}{\lambda_*}U_+(\xi+\tfrac{1}{2}),$$

this final equation being obtained from the degenerate sum of the horizontal momentum equations, which are linearly dependent at lowest order. The solution is most readily interpreted when expressed in terms of Fourier constituents, e.g.

$$\overline{u}_n = \int_{-1}^{0} \overline{u}\,e^{-2\pi n i\xi}\,d\xi.$$

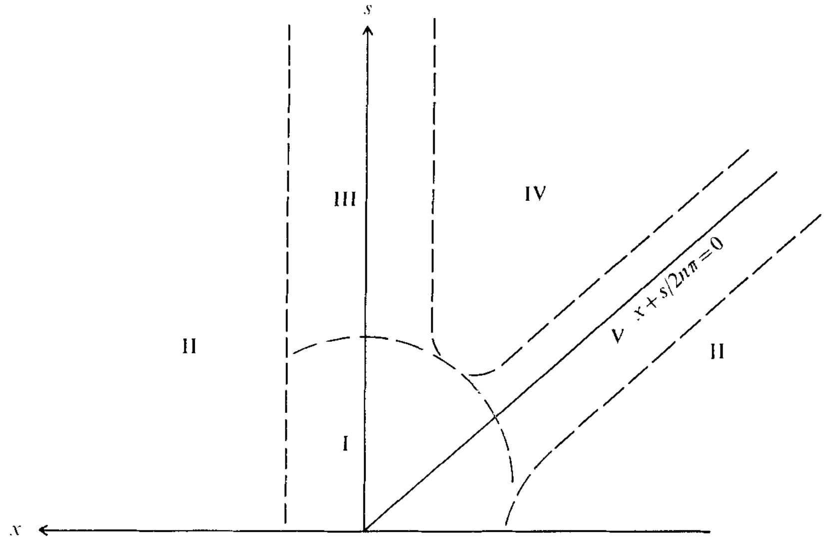

FIGURE 2. Regions extending the $O(\epsilon^{\frac{1}{2}})$ (θ, τ) neighbourhood of a critical latitude.

The initial condition $\overline{u} = 0$ $(s = 0)$ and boundary conditions $\overline{u} \to 0$ $(y \to \pm\infty)$ and $\overline{w} = 0$ $(\xi = -1, 0)$ yield

$$\overline{u}_0 = 0, \quad \overline{u}_n = \frac{mU_+}{2\lambda_*}\exp(-\pi nix^2)\int_x^{x+s/2\pi n}\exp(\pi niy^2)\,dy \quad (n \neq 0), \tag{4.1}$$

to which the other quantities are related by

$$\overline{v}_n = \overline{u}_n, \quad \overline{p}_n = \frac{\overline{u}_n}{2\pi ni} \quad (n \neq 0), \quad \overline{w}_n = \frac{1}{2\pi ni}\frac{\partial \overline{u}_n}{\partial x} \quad (n \neq 0),$$

$$\overline{p}_0 = \sum_{n \neq 0} \overline{p}_n, \quad \overline{w}_0 = -\sum_{n \neq 0} \overline{w}_n.$$

For moderate x and s,

$$\overline{u}_n \sim \frac{imU_+}{4\pi nx\lambda_*}(1-e^{ixs}) + O\left(\frac{1}{n^2}\right) \quad \text{as} \quad n \to \infty,$$

so that

$$\overline{u} - (\xi + \tfrac{1}{2})\frac{mU_+}{2x\lambda_*}(1-e^{ixs})$$

has an absolutely convergent Fourier series of terms $O(1/n^2)$ $(n \to \infty)$.

The form (4.1) may be expressed in terms of Fresnel integrals, for which limiting forms are known for large or small x and $x+s/2\pi n$ (Abramowitz & Stegun 1965, p. 300). For small $s = \delta T$, $\delta\overline{u}_n \sim \epsilon mU_+T/(4\pi n)$ matches the Fourier constituent of the initial solution $\epsilon U_1 \sim -\frac{1}{2}\epsilon imU_+T(\xi+\frac{1}{2})$ at large T. For large x or s (see figure 2), (4.1) implies the existence of Fresnel wave fronts along $x = O(1)$ (region III) and $x+s/2\pi n = O(1)$ (region V) for each vertical structure mode n. In order to follow the solution for longer times, the regions II–V demarcated by these fronts must be considered separately. Region II (large $|x|$, given s) corresponds to large distances from critical latitudes. The asymptotic form of (4.1) agrees with the solution for region II developed to $T = O(\epsilon^{-1})$ in §3.

4.2. *Later times*

Region III is a $\epsilon^{\frac{1}{2}}$ neighbourhood of the critical latitude. We divide the integral ranges of (4.1) into $(\infty, x+s/2\pi n)$ and (x, ∞). The former becomes small as $s \to \infty$, and when $\tau \equiv \epsilon T = O(1)$ it contributes motion of the form

$$\epsilon \mathbf{U}^{(1)}(\xi, \tau) \exp[i\phi_n(\tau)/\epsilon],$$

which may be identified with the transient inertial wave discussed in §3. The major contribution comes from the range $x \leqslant y < \infty$, which represents a standing Fresnel wave front at the critical latitude $\mu = \lambda$. Although obtained in §4.1 as a solution for $T = O(\epsilon^{-\frac{1}{2}})$, this part of (4.1) is clearly independent of time, and therefore persists indefinitely as a valid solution. We return to this after considering the related motion in the two remaining regions.

For region IV we may again identify the range $(\infty, x+s/2\pi n)$ in (4.1) as the transient inertial wave which develops according to §3. The range $x \leqslant y < \infty$ contributes

$$\frac{mU_+}{2\lambda_*(2n)^{\frac{1}{2}}}(1+i)\exp(-n\pi i x^2)$$

to (4.1), and when $\tau \equiv \epsilon T = O(1)$ may be expected to develop as

$$(U, V, \epsilon W, \epsilon^{-1}P) = \epsilon^{\frac{1}{2}} \mathbf{U}^{(\frac{1}{2})}(\tau, \mu, \xi) \exp[i\phi_n(\tau, \mu)/\epsilon],$$

which takes account of the rapid variations in $x \equiv (\mu - \lambda)/\lambda_* \delta$ away from the critical latitude. We expect $\partial\phi_n/\partial\tau = \lambda$ in order to retain the time factor $e^{i\lambda T}$. Since this form is the same as (3.1) for the transient inertial waves, apart from the greater magnitude $\epsilon^{\frac{1}{2}}$, the same equations govern its evolution. Thus region IV contains $O(\epsilon^{\frac{1}{2}})$ inertial waves developing along group-velocity characteristics (as described in §3) which emerge from region III (region V is the $\epsilon^{\frac{1}{2}}$ neighbourhood of the characteristic starting at $\mu = \lambda$). Since the frequency σ is λ at the region III source of the $O(\epsilon^{\frac{1}{2}})$ motion, the characteristics all have $\sigma = \lambda$ (i.e. $\theta_0 = \theta_c$), and are distinguished by their starting time $\tau_0 > 0$ at $\theta = \theta_c$. Replacing (3.5), we therefore have

$$0 = \left(\frac{\tau - \tau_0}{n\pi}\right) \sin\theta_c \sin^2 2\theta_c + \{2(\theta_c - \theta) + \cos 2\theta_c(\sin 2\theta - \sin 2\theta_c)\}$$

to describe the group-velocity characteristics of the $O(\epsilon^{\frac{1}{2}})$ inertial waves in region IV. The characteristics are all identical with that of figure 1 which begins at the critical latitude $\mu = \lambda$, apart from the displacement τ_0 in τ.

Region V forms a transition when $T = O(\epsilon^{-1})$ between the advancing region IV of $O(\epsilon^{\frac{1}{2}})$ inertial waves of frequency λ coming from the critical latitude, and the region II unaffected by critical-latitude phenomena and containing merely the $O(\epsilon)$ transient inertial wave. A solution of the following form may be sought:

$$(U, V, \epsilon W, \epsilon^{-1}P) = \delta(\mathbf{U}^{(\frac{1}{2})} + \delta\mathbf{U}^{(1)} + \delta^2\mathbf{U}^{(\frac{3}{2})} + \dots)(y, \xi, \tau; \theta) \exp[i\lambda T + i\phi_n(\theta)/\epsilon],$$

where $\delta \equiv \epsilon^{\frac{1}{2}}$, and $y \equiv (\theta - \theta_F(\tau))/\delta$ measures distance from the group velocity characteristic $\theta = \theta_F(\tau)$ starting at $\theta = \theta_c$, $\tau = 0$; the length scale δ is suggested by the $T = O(\epsilon^{-\frac{1}{2}})$ solution. $\phi_n(\theta)$ is the known phase of the region IV inertial waves. Substituting in (2.2) and proceeding to relative order δ^2, an equation governing the form of $\mathbf{U}^{(\frac{1}{2})}$ is eventually obtained. This is found to admit a similarity solution, namely

$$W^{(\frac{1}{2})} = f(\tau)\, Y(g(\tau)\, y),$$

where $$Y(z) = \int_{\infty}^{z} \exp\left(\tfrac{1}{2}\beta x^2\right) dx \quad (\beta \text{ is a positive constant})$$

represents a Fresnel-integral wave front advancing along $\theta_F(\tau)$. The width $1/g(\tau)$ of the wave front varies as the front progresses, i.e.

$$\beta g^2 = \frac{n\pi}{\lambda^2\lambda_*^2} \frac{(1-\cos 2\theta_c \cos 2\theta_F)^2}{\partial F/\partial\theta_c},$$

and the amplitude $f(\tau)$ matches that of the $O(\epsilon^{\frac{1}{2}})$ inertial waves of region IV. This solution also matches the $O(\epsilon)$ transient inertial waves of region II.

To summarize the above picture, we have found that at the critical latitude the vertical component $-\epsilon\mu_* V_0$ of the Coriolis force on the $O(1)$ Haurwitz normal mode is balanced by a pressure correction ϵP_1. Having frequency λ, this causes a near-resonant inertial-wave response at the critical latitude $\mu = \lambda$. Inertial waves $O(\epsilon^{\frac{1}{2}})$ of all vertical wavenumbers n are generated, each at a standing Fresnel wave front (region III) of width $\epsilon^{\frac{1}{2}}$ at the critical latitude. The energy of each mode n propagates away (southwards for $n > 0$, northwards for $n < 0$) with its group velocity to fill the spreading region IV behind an advance Fresnel wave front, region V.

5. Discussion

We have considered the motion in a thin (thickness ϵ) spherical shell evolving from an initial state in which the velocity corresponds to one of the normal-mode solutions of LTE found by Haurwitz (1940). Any initial velocity field with the global length scale is expressible as a linear combination of these, so that no loss of generality is incurred by considering one mode individually. The dominant feature of the subsequent motion is the normal-mode solution P_n^m of LTE, with frequency $[2\Omega m/n(n+1)] + O(\epsilon^2)$.

The largest additions to the LTE normal mode are $O(\epsilon^{\frac{1}{2}})$ inertial-wave fine-structure modes of frequency $2\Omega m/n(n+1)$, generated at and propagating from critical latitudes as found in §4. Since they spread from the critical latitudes at a rate $\Omega\epsilon$, behind the various Fresnel wave fronts corresponding to the separate vertical structure modes, they represent an energy loss rate $O(\Omega\epsilon^2)$ from the driving LTE normal mode. This energy transfer continues indefinitely (implying a decay rate $O(\Omega\epsilon^2)$ for the Haurwitz normal-mode form), and is perhaps the most important factor in any discussion of the 'validity' of the normal modes.

It appears that any 'normal mode' must have fine-scale inertial-wave velocities of order $\epsilon^{\frac{1}{2}}$ which are not square-integrable, on account of the infinite energy transfer to such motions which is implied by the indefinite maintenance of the LTE normal-mode oscillation. Hence the SR solution is as well behaved as possible. However, we argue that the $O(\epsilon^{\frac{1}{2}})$ terms in the solution of the initial-value problem do not tend to those of the SR modified normal mode in any meaningful sense. Indeed, the most realistic approach is probably to ignore the transient inertial waves, and to include the $O(\epsilon^{\frac{1}{2}})$ inertial waves, generated at critical latitudes, only to the extent of that part

$$\bar{u}_n = \frac{mU_+}{2\lambda_*} \exp(-\pi n i x^2) \int_x^{\infty} \exp(\pi n i y^2)\, dy$$

of (4.1) which continues indefinitely. In doing this we neglect the repeated returns of the advancing wave front (region V) after passage through a pole of rotation. Such an assumption is appropriate if the fine-structure inertial motion is somehow dissipated on its passage around the spherical shell, and corresponds to the radiation condition that all the fine-structure motion should radiate only away from its critical latitude source. This contrasts with the SR fine structure, in which equal radiation towards and away from the critical latitudes in each mode is implied by the form SW(6.1). Even without appeal to dissipation of the fine structure as it passes around the shell, we recall (cf. § 3) that inertial waves pass smoothly through their critical latitude. Hence there is *net* energy radiation from the critical latitudes even with their return. Their essentially random phase relation on return after $O(\epsilon^{-1})$ oscillations (especially as $\epsilon \rightarrow 0$) also makes inclusion of the returning waves inappropriate. If in fact we do appeal to dissipation, the $O(\epsilon^{\frac{1}{2}})$ inertial motions of course never build up to the point where they cease to be integrable in square. Indeed, with or without dissipation, a velocity field not integrable in square, albeit at $O(\epsilon^{\frac{1}{2}})$, cannot be approached by a sensible representation of a finite-energy solution.

In the context of an initial-value problem, the Haurwitz normal modes do appear to be useful in that, until times $O(\Omega^{-1}\epsilon^{-2})$, they may be combined as indicated by the initial conditions to represent the bulk of the energy in the evolving motion. After such times, however, the energy in the fine-structure motion (which is essentially chaotic as $\epsilon \rightarrow 0$) has become appreciable. The introduction of kinematic viscosity ν does not affect this conclusion unless $\nu \gtrsim \Omega a^2 \epsilon^6$, in which case the viscous decay rate $\Omega(\nu/\Omega a^2 \epsilon^2)^{\frac{1}{2}}$ of both the normal modes and the fine structure exceeds the energy transfer $\Omega\epsilon^2$. Forced oscillations are another case in which normal modes may form a useful representation. In this context we may expect continual generation, at the forcing frequency σ, of fine-scale inertial wave motions radiating from the corresponding critical latitudes. At non-resonant frequencies ($\sigma \neq 2\Omega m/n(n+1)$), the Haurwitz normal modes then form a valid representation of the motion only if we again suppose that the fine structure is somehow dissipated in its passage around the sphere. Otherwise, an indefinite build-up of the inertial wave motion eventually swamps the constituents of global length scale.

Perhaps the most unsatisfactory feature of the analysis is the decomposition into an infinite set of vertical structure modes, introducing questions of convergence. However, the wave fronts found, having greater speeds for lower modes, appear to form a potentially observable feature of the individual modes. Any numerical calculation involving a finite-difference scheme inevitably truncates the system to a finite number of such modes, and can therefore merely check and not supplement this analysis. The results of numerical analysis will depend crucially on the choice of radiation condition (e.g. inertial-wave radiation only away from critical latitudes). This is an input rather than a result of the calculation.

Experiments (Aldridge 1973) have generally been concerned with the resonant frequencies and form of axisymmetric oscillations between concentric spheres with $\epsilon = O(1)$ (rather than $\epsilon \ll 1$ as for the thin shell). There is no general corresponding theory; the conclusions appear to be that, while observed resonant frequencies correspond to numerical values based on a Rayleigh quotient (Aldridge 1973), the form of oscillation is complicated. The thin shell $\epsilon \ll 1$, regarded as a first step towards $\epsilon = O(1)$, is in broad agreement with these observations, as far as either goes, in predicting an enhanced response at the natural-mode frequencies $m/n(n+1)$ with continued energy transfer to fine scales.

Hitherto we have discussed the 'validity' of normal modes derived from LTE. From a more practical (e.g. oceanographic) viewpoint, perhaps the chief features of interest in the solution to the initial-value problem are (i) the continual generation of $O(\epsilon^{\frac{1}{2}})$ inertial waves at critical latitudes and (ii) the manner in which free inertial waves evolve and cover the sphere.

Section 4 implies that any sinusoidal oscillation of frequency σ less than 2Ω and global length scale generates inertial waves at its critical latitudes ($\cos\theta_c = \pm\sigma/2\Omega$). The energy transfer is at a rate $O(\Omega\epsilon^2)$ relative to the global energy in the forcing oscillation, and the inertial waves, of relative amplitude $O(\epsilon^{\frac{1}{2}})$, radiate this energy away from the critical latitudes at a speed $O(a\Omega\epsilon)$. In doing so they develop $O(\epsilon a)$ horizontal wavelengths but retain the forcing frequency.

The results (ii) are described in §3 for the $O(\epsilon)$ transient inertial oscillations of the initial-value problem. However, some apply to all free inertial waves of given azimuthal wavenumber m. Thus they retain their frequency and vertical structure number n (not precisely the vertical wavenumber) while their meridional wavenumber is determined by the local dispersion relation (§3.1). Their energy travels with the group velocity of the particular vertical mode, i.e. along the characteristics shown in figure 1 and its mirror image. Hence the amplitude and phase of the wave envelope evolve as stated (for $W(\mu, \tau)$) in §3.2.

In the initial-value problem there was an $O(\epsilon)$ initial distribution of horizontal motion having the local inertial frequency everywhere. From any such beginning, satisfaction of the local dispersion relation as the energy propagates implies a gain of vertical motion and small $O(\epsilon a)$ horizontal scales. Waves from the northern hemisphere progressing south form caustics, or fronts of greatest advance, in the form of an Airy function with a relative amplitude gain $\epsilon^{-\frac{1}{6}}$. The fronts span a breadth $a\epsilon^{\frac{2}{3}}$. One forms for each vertical mode, those for the lowest modes travel fastest, and there are corresponding fronts in the northern hemisphere marking the greatest advance of waves from the south. In passing across the poles, inertial waves suffer a phase change of $\frac{1}{2}\pi$.

This problem was suggested to me by Professor J. W. Miles, and I should also like to thank Professor K. Stewartson and the referees for helpful comments on an earlier version. The work was begun at the Institute of Geophysics and Planetary Physics, University of California at San Diego, with the partial support of the Atmospheric Sciences Section, National Science Foundation, N.S.F. Grant GA-35396 X.

REFERENCES

ABRAMOWITZ, M. & STEGUN, I. A. 1965 *Handbook of Mathematical Functions*. Dover.

ALDRIDGE, K. D. 1973 Axisymmetric inertial oscillations of a fluid in a rotating spherical shell. *Mathematika* **19**, 163–168.

HAURWITZ, B. 1940 The motion of atmospheric disturbances on the spherical earth. *J. Mar. Res.* **3**, 254–267.

LONGUET-HIGGINS, M. S. 1964 Planetary waves on a rotating sphere. *Proc. Roy. Soc.* A **279**, 446–473.

MILES, J. W. 1974 On Laplace's tidal equations. *J. Fluid Mech.* **66**, 241–260.

PEKERIS, C. L. 1975 A derivation of Laplace's tidal equation from the theory of inertial oscillations. *Proc. Roy. Soc.* A **344**, 81–86.

STEWARTSON, K. 1971 On trapped oscillations of a rotating fluid in a thin spherical shell. *Tellus* **23**, 506–510.

STEWARTSON, K. & RICKARD, J. A. 1969 Pathological oscillations of a rotating fluid. *J. Fluid Mech.* **35**, 759–773.

STEWARTSON, K. & WALTON, I. C. 1976 On waves in a thin shell of stratified rotating fluid. *Proc. Roy. Soc.* A **349**, 141–156.

WALTON, I. C. 1975 On waves in a thin rotating spherical shell of slightly viscous fluid. *Mathematika* **22**, 46–59.